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LETTER TO THE EDITOR

Cumulant renormalisation group and its application to the incipient infinite cluster in percolation

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Abstract. We introduce the cumulant method into the context of position-space renormalisation group. The advantage of this method over previous procedures is that one need not systematically evaluate all 2^N states of each Kadanoff cell, where N is the number of elements (sites or bonds) in the cell. We illustrate this method by calculating, for bond percolation with an eight-bond cell, the critical exponents characterising the following quantities: (i) the mean number of backbone bonds, (ii) the mean number of bonds in the minimum and maximum paths along the backbone, (iii) the mean number of singly connected ('red') bonds, and (iv) the mean number of bonds in all the self-avoiding walks connecting the two extreme points. We also derive previously obtained results for the conductivity and resistivity.

Position-space renormalisation group (PSRG), in all its various forms and variations, has been the object of much investigation since its inception a decade ago (see e.g. Burkhardt and van Leeuwen 1982). One reason for the success of PSRG is that it has served an important complementary role to conventional momentum-space RG work, which generally is most successful at, above, and somewhat below the upper marginal dimensionality d_c. For d well below d_c, however, the basic assumption that 'weight functions' are roughly Gaussian breaks down (Bruce 1981, Binder 1981) and it is here that the predictions of PSRG are often more reliable.

In the general area of connectivity phenomena (such as percolation, linear polymers and branched polymers), PSRG has played a particularly important role (see e.g. Stanley *et al* (1982) and references therein). Among the reasons for this is the fact that, at least until recently, there were no exact results for critical properties even for $d = 2$ (with the exception of the exact values for the percolation threshold p_c for a few lattices with special properties).

One advantage of PSRG approaches to connectivity phenomena is their *directness*: no Hamiltonian is, in general, necessary. Hence the approach is quite physical provided one chooses a reasonable renormalisation transformation that adequately reflects the connectivity of the problem under study. On the other hand, a serious disadvantage of PSRG approaches is that the calculations are generally quite complicated since one must in general enumerate all 2^N configurations of an N-element Kadanoff cell (where, e.g., the elements are bonds that can be intact or broken). Thus only the smallest cells can be treated exactly, and the predictions of small cells are often not too accurate.

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The 'trend' for increasing cell size is, however, very regular and therefore extrapolations based on results from a sequence of ever-increasing cell sizes can be quite reliable (Reynolds *et al* 1980). Thus any method that can reduce the total number of diagrams that must be calculated is of potential utility.

The basic assumption underlying PSRG is 'self-similarity': the essential physics of the system is not influenced by a scale change. For example, if some percolation quantity $Q(p)$ diverges at p_c with exponent x,

$$Q(p) \sim |p - p_c|^x, \tag{1a}$$

then under the RG transformation, the new quantity $Q'(p')$ also diverges with the same exponent,

$$Q'(p') \sim |p' - p_c|^x, \tag{1b}$$

where $p' = R(p)$ is the renormalised probability after a renormalisation transformation. If the effect of the scale change on $Q(p)$ is $Q' = \lambda_Q Q$, then (1*a*) and (1*b*) imply

$$x = \ln \lambda_Q / \ln \lambda_p, \tag{2}$$

where $\lambda_p = dR(p)/dp|_{p=p^*}$, where p^* is the fixed point of the RG transformation. For example, if $Q(p)$ is the correlation length $\xi(p)$, then $\lambda_Q = 1/b$. Thus the exponent of any quantity $Q(p)$ can be calculated if we can calculate the recursion relation in the vicinity of the fixed point. It is to this task that we now turn our attention.

We illustrate the essential idea of cumulant renormalisation group (CRG) by means of an example: a 2×2 bond cell on the square lattice (figure 1(*a*)). This cell is renormalised into a single bond. If we are interested in properties of the backbone, then we imagine that the extreme east bonds are connected by a bus bar, and similarly for the extreme west bonds. Then the 2×2 cell is equivalent to a Wheatstone bridge (figure 1(*b*)).

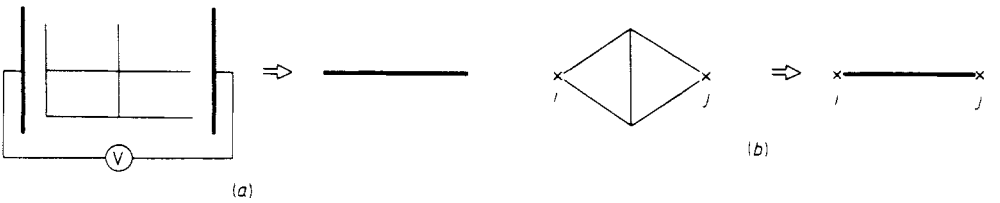

Figure 1. (*a*) An eight-bond Kadanoff cell of Reynolds *et al* (1977) for bond percolation on a square lattice with length rescaling parameter $b = 2$. (*b*) The corresponding five-bond Wheatstone bridge obtained when one places bus bars on the east and west extremities of part (*a*).

The question to be addressed is 'what is the recursion relation $p' = R(p)$ giving the probability of the *renormalised* bond to be intact, where p is the probability of the *original* bonds to be intact?' Generally, for any quantity Q we have (Stanley *et al* 1982)

$$\langle Q \rangle = \sum_{2^8 \text{ configurations}} Q(\text{config})\, p(\text{config}), \tag{3}$$

where $Q(\text{config})$ is the value of the quantity in one of the 2^8 configurations of the bond cell, and $p(\text{config}) = p^b(1-p)^{8-b}$ is the probability of the given configuration of

b intact bonds. As an example, we have for the number of intact 'backbone' bonds (Shlifer *et al* 1979)

$$\chi_B(p) = 5p^5 + p^4q(5\times 4) + p^3q^2(3\times 2 + 2\times 6) + p^2q^3(2\times 2) + pq^4(5\times 0) + q^5(0). \tag{4}$$

In writing (4), we have systematically evaluated the 2^5 configurations of the backbone bonds; the dangling ends are not considered.

The same results can be obtained by using the cumulant method, according to which for any graph G and any quantity Q, the average Q_{ij} between points i and j of figure 1(b) is given by

$$Q_{ij} = \sum_g C(g)E(g)p^{b(g)}, \tag{5}$$

where the summation runs over all subgraphs, which are constructed by SAW paths, of the original graph G, $E(g)$ is the embedding constant, $b(g)$ is the number of bonds in g, and the cumulant $C(g)$ satisfies the recursive relation

$$C(g) = V(g) - \sum_{g'} C(g'). \tag{6}$$

Here $V(g)$ is the value of g, while g' are the subgraphs of g.

If we apply this method to the Wheatstone bridge of figure 1(b),

$$\langle \text{  } \rangle_{av} = C(\text{})p^5 + 4C(\text{})p^4 + C(\text{})p^4 + 2C(\text{})p^3 + 2C(\text{})p^2$$

$$= (V(\text{}) - 4V(\text{}) - V(\text{}) + 2V(\text{}) + 4V(\text{}))p^5$$

$$+ 4[V(\text{}) - V(\text{}) - V(\text{})]p^4$$

$$+ [V(\text{}) - 2V(\text{})]p^4 + 2V(\text{})p^3 + 2V(\text{})p^2. \tag{7}$$

To test the CRG, we first calculate the probability that sites i and j are connected. From (7), we readily find

$$R(p) = 2p^2 + 2p^3 - 5p^4 + 2p^5. \tag{8a}$$

This agrees with the recursion relation calculated by Reynolds *et al* (1977). Also we can calculate the mean resistance between sites i and j, where each bond is taken to have resistance unity. We find

$$\chi_R = 4p^2 + 6p^3 - \tfrac{49}{3}p^4 + \tfrac{22}{3}p^5, \tag{8b}$$

which agrees with Bernasconi (1978).

Similarly we can calculate the mean number of backbone bonds. Actually there are two physically plausible definitions of the backbone bonds between points i and j. In definition 1, we follow Shlifer *et al* (1979) and define a bond to be part of the backbone if it belongs to the intersection of the set of all self-avoiding walks between points i and j. In definition 2, we follow customary intuition and define a bond to belong to the backbone if it carries current when a potential difference is applied between points i and j. Thus, e.g., the central bond of the Wheatstone bridge of figure 1(b) is considered a backbone bond by definition 1, but not by definition 2. For the mean number of backbone bonds, we find the new result

$$\chi_{BB1} = 4p^2 + 6p^3 - 4p^4 - p^5, \qquad \chi_{BB2} = 4p^2 + 6p^3 - 4p^4 - 2p^5. \tag{8c, d}$$

If we set $q = 1 - p$ in (4), we recover (8c).

We find for the mean number of bonds in the minimum and maximum paths along the backbone, respectively,

$$\chi_{\min} = 4p^2 + 6p^3 - 14p^4 + 6p^5, \qquad \chi_{\max} = 3p^5 - 10p^4 + 6p^3 + 4p^2. \tag{8e,f}$$

The structure of the incipient infinite cluster has been proposed to consist of a backbone and numerous dangling ends; the structure of the former is a sequence of singly connected 'red' bonds interrupted by multiply connected 'blue' bonds (Stanley 1977, Coniglio 1981, 1982). If we calculate the mean number of 'red' bonds, we find

$$\chi_{\text{red}} = 4p^2 + 6p^3 - 20p^4 + 10p^5. \tag{8g}$$

Notice that $\chi_{\text{red}} = p(\mathrm{d}R/\mathrm{d}p)$ so that the Coniglio (1982) theorem holds for the Wheatstone bridge. If we calculate the mean number of bonds in all the self-avoiding walks connecting the two extreme points of figure 1(*c*), we find

$$\chi_{\text{MSAW}} = 4p^2 + 6p^3 - 12p^4 + \tfrac{9}{2}p^5. \tag{8h}$$

Since the eight-bond cell of figure 1(*a*) is self-dual, $p^* = \frac{1}{2}$, which is the exact value for bond percolation on a square lattice. Hence $\lambda_p = \frac{13}{8}$ from (8*a*). The renormalised cell has only one bond (figure 1(*a*)), so the renormalised values for *all* the above quantities are simply p^* at $p = p^*$. Therefore λ_Q for each quantity is obtained by dividing the values of the functions of equations (8) by p^*. Thus we find $\lambda_{\text{R}} = \frac{92}{48}$, $\lambda_{\text{BB1}} = \frac{47}{16}$, $\lambda_{\text{BB2}} = \frac{23}{8}$, $\lambda_{\min} = \frac{17}{8}$, $\lambda_{\max} = \frac{39}{16}$ and $\lambda_{\text{red}} = \frac{13}{8}$ and $\lambda_{\text{MSAW}} = \frac{73}{32}$.

The corresponding critical exponents are given by substituting these values into (2). Thus for the behaviour of the effective one-dimensional resistance $L_{\text{R}} \sim \varepsilon^{-\zeta_{\text{R}}}$ we have

$$\zeta_{\text{R}} = 1.34, \tag{9a}$$

where $\varepsilon = (p_{\text{c}} - \text{p})/p_{\text{c}}$. Similarly for the two different definitions of the backbone, we find $L_{\text{BB1}} \sim \varepsilon^{-\zeta_{\text{BB1}}}$ and $L_{\text{BB2}} \sim \varepsilon^{-\zeta_{\text{BB2}}}$, where

$$\zeta_{\text{BB1}} = 2.219, \qquad \zeta_{\text{BB2}} = 2.175. \tag{9b}$$

For the mean number of bonds in the minimum and maximum length paths between sites i and j, we define $L_{\min} \sim \varepsilon^{-\zeta_{\min}}$ and $L_{\max} \sim \varepsilon^{-\zeta_{\max}}$ and find

$$\zeta_{\min} = 1.55, \qquad \zeta_{\max} = 1.835. \tag{9c}$$

Since $\lambda_{\text{red}} = \lambda_p$, it follows that

$$\zeta_{\text{red}} = 1, \tag{9d}$$

where ζ_{red} is defined through $L_{\text{red}} \sim \varepsilon^{-\zeta_{\text{red}}}$, and L_{red} is the mean number of singly connected bonds between i and j. This is in accord with the Coniglio theorem that $\zeta_{\text{red}} = 1$ for all d. Finally, for the mean number of bonds in all the self-avoiding walks between i and j, we define $L_{\text{MSAW}} \sim \varepsilon^{-\zeta_{\text{MSAW}}}$, and find

$$\zeta_{\text{MSAW}} = 1.699. \tag{9e}$$

These results are compared with independent calculations by other methods in table 1, and the degree of agreement is surprising considering the simplicity of the present approach and the relatively small size of the Kadanoff cell used.

In summary, then, we have introduced the cumulant method into PSRG and found that it substantially reduces the labour necessary for the explicit calculations required.

Table 1. Comparison between exponents calculated by cumulant renormalisation group for the 2×2 bond cell of figure 1(*a*) and predictions for the same exponents obtained by other methods.

Exponent	Cumulant renormalisation group	Other methods
ζ_R	1.34	1.43[a], 1.1[b], 1.28[c]
ζ_{BB1}	2.22	2.4[d]
ζ_{BB2}	2.18	not calculated before
ζ_{min}	1.55	1.4[d], 1.49[e]
ζ_{max}	1.835	not calculated before
ζ_{red}	1.000	1.0[d,e,f]
ζ_{MSAW}	1.699	not calculated before

[a] Fisch and Harris (1978).
[b] Stinchbombe and Watson (1976), Harris and Kirkpartick (1977).
[c] Derrida and Vannimenus (1982).
[d] Hong and Stanley (1983).
[e] Pike and Stanley (1981).
[f] Coniglio (1982).

We plan to apply this method to the 3×3 Kadanoff cell, for which the corresponding generalisation of the five-bond Wheatstone bridge has 13 bonds. This is by itself a non-trivial task, and results will be reported in a future work.

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